Calculus II - Day 3

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Goals for Today

- Define what it means for a series to converge
- Determine when geometric series converge and find the sum
- Find sums of ”telescoping” series

Homework

- Gradescope HW #1: due Tuesday evening
- MyLab HW #2: due Friday at noon

A series is denoted by:

$$\sum_{k=1}^{\infty} a_k$$

which represents the sum of the terms in a sequence $\{a_k\}_{k=1}^{\infty}$.
Sequence:

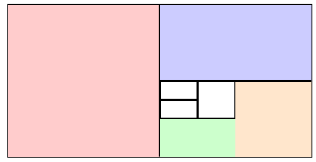
$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$$

Series:

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots$$

Question: How is it possible to add infinitely many numbers and get something finite?
Example of a ”convergent” series:

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$$



How do we determine whether a series converges or diverges: that is, whether

$$\sum_{k=1}^{\infty} a_k = \text{some number ?}$$

Definition: Let $\{a_k\}_{k=1}^{\infty}$ be a sequence.

The Nth partial sum S_N is the finite sum obtained by adding the first N terms of the sequence:

$$S_N = a_1 + a_2 + \cdots + a_N = \sum_{k=1}^{N} a_k$$

This creates a new sequence $\{S_N\}_{N=1}^{\infty}$:

$$\{S_1, S_2, S_3, \dots\} = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots\}$$

(versus $\{a_n\}_{k=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$)

If the sequence $\{S_N\}$ converges:

$$\lim_{N\to\infty} S_N = S$$

then we say the series $\sum_{k=1}^{\infty} a_k$ converges, and the sum is S.

Otherwise, the series diverges.

Sum of series = limit of sequence of partial sums

Example:

$$\sum_{k=0}^{\infty} \frac{1}{2^k} \quad \left(\text{where } a_k = \frac{1}{2^k},\ k = 0, 1, 2, \dots\right)$$

$$S_1 = a_0 = 1$$

$$S_2 = a_0 + a_1 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$S_3 = a_0 + a_1 + a_2 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

$$S_4 = a_0 + a_1 + a_2 + a_3 = \frac{15}{8}$$

Thus, the sequence of partial sums is:

$$\{S_N\}_{N=1}^{\infty} = \left\{\frac{1}{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \dots\right\}$$

What's the formula?

$$S_N = 2 - \frac{1}{2^{N-1}} = \frac{2^N - 1}{2^{N-1}}$$

Does the partial sum sequence converge?

$$\lim_{N\to\infty}\left(2 - \frac{1}{2^{N-1}}\right)$$

$$= 2 - \lim_{N\to\infty}\frac{1}{2^{N-1}} = 2 - 0 = \boxed{2}$$

If we can find a formula for S_N, determining whether $\sum a_k$ converges becomes a problem about sequences!

Definition: Let a and r be real numbers, $a \neq 0$, $r \neq 1$. The series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$$

is called a geometric series, with common ratio r.

Nth partial sum of a geometric series:

$$S_1 = a$$

$$S_2 = a + ar$$

$$S_3 = a + ar + ar^2$$

$$\vdots$$

$$S_N = a + ar + ar^2 + \cdots + ar^{N-1}$$

$$S_N = \sum_{k=0}^{N-1} ar^k$$

$$S_N = a + ar + ar^2 + ar^3 + \cdots + ar^{N-1}$$

$$rS_N = ar + ar^2 + ar^3 + \cdots + ar^{N-1} + ar^N$$

$$\begin{array}{rcl} S_N & = & a + ar + ar^2 + \cdots + ar^{N-1} \\ -rS_N & = & -(ar + ar^2 + \cdots + ar^N) \\ \hline S_N - rS_N & = & a - ar^N \end{array}$$

$$S_N(1-r) = a(1-r^N) \quad \Rightarrow \quad \boxed{S_N = \frac{a(1-r^N)}{1-r}}$$

Partial sum formula for geometric series

When does $\lim\limits_{N\to\infty} \frac{a(1-r^N)}{1-r}$ converge?

It converges exactly when $|r| < 1 \quad (-1 < r < 1)$.
When this is the case:

$$\lim_{N\to\infty} \frac{a(1-r^N)}{1-r} = \boxed{\frac{a}{1-r}}$$

Otherwise, the series diverges.

Example:

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \sum_{k=0}^{\infty} 1 \cdot \left(\frac{1}{2}\right)^k$$

is geometric with $a = 1$, $r = \frac{1}{2}$.

$$\Rightarrow \text{converges to } \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

Determine whether the following geometric series converge or diverge. If it converges, find the limit.

a) $2 + \frac{4}{10} + \frac{8}{100} + \frac{16}{1000} + \dots$
It approaches 2.5.

$$\sum_{k=0}^{\infty} 2\left(\frac{2}{10}\right)^k \Rightarrow \frac{2}{1-\frac{2}{10}} = 2.5$$

b) $-5 + 5 - 5 + 5 - 5 + \dots$

It jumps from -5 to 0 forever and doesn't converge on anything, diverges.

c) $3 - \frac{6}{5} + \frac{12}{25} - \frac{24}{125} + \dots$

$$\sum_{k=0}^{\infty} 3\left(-\frac{2}{5}\right)^k \Rightarrow \frac{3}{1-\left(-\frac{2}{5}\right)} = \frac{3}{1+\frac{2}{5}} = \frac{3}{\frac{7}{5}} = 3 \cdot \frac{5}{7} = \boxed{\frac{15}{7}}$$

Example:

$$\sum_{k=1}^{\infty} 4\left(\frac{1}{3}\right)^k = \sum_{k=1}^{\infty} 4 \cdot \frac{1}{3} \cdot \left(\frac{1}{3}\right)^{k-1}$$

$$= \sum_{k=0}^{\infty} \frac{4}{3} \cdot \left(\frac{1}{3}\right)^k = \frac{\frac{4}{3}}{1-\frac{1}{3}} = \frac{\frac{4}{3}}{\frac{2}{3}} = 2$$

Problem: Generally difficult to find a formula for S_N.
Sometimes cancellation can help us:
Example:

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$S_1 = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$$

$$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2}{3}$$

$$S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{3}{4}$$

$$S_4 = \cdots = 1 - \frac{1}{5} = \frac{4}{5}$$

In general:

$$S_N = \sum_{k=1}^{N} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

$$S_N = 1 - \frac{1}{N+1}$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1$$

This is called a telescoping series.

For Monday:

$$\sum_{k=2}^{\infty} \left(\sin\left(\frac{\pi}{k}\right) - \sin\left(\frac{\pi}{k+1}\right)\right)$$